On CR-Structure And F-Structure Satisfying

$F^{p_1p_2}+F=0$

Lakhan Singh, Shailendra Kumar Gautam

***Abstract*— In this paper, we have studied a relationship between CR-structure and F-structure satisfying $F^{p_1p_2}$ + F=0, where p_1 and p_2 are twin primes. Nijenhuis tensor and integrability conditions have also been discussed.**

***Index Terms*— Projection operators, distributions, Nijenhuis tensor, integrability conditions and CR-structure.**

I. Introduction

Let M be an n-dimensional differentiable manifold of class C^∞. Let F be a non-zero tensor of type (1, 1) and class C^∞ defined on M, such that

1.1 $$F^{p_1p_2}+F=0$$

where p_1 and p_2 are twin primes.

Let rank $((F))=r$, which is constant everywhere. We define the operators on M as

1.2 $$l=-F^{p_1p_2-1},\ m=F^{p_1p_2-1}+I$$

where I is the identity operator on M.

Theorem (1.1) Let M be an F- structure satisfying (1.1) Then

(1.3) (a) $$l+m=I$$

(b) $$l^2=l$$

(c) $$m^2=m$$

(d) $$lm=ml=0$$

Proof: From (1.1) and (1.2), we get the results.

Let D_l and D_m be the complementary distributions corresponding to the operators l and m respectively. then

$$\dim((D_l))=r,\quad \dim((D_m))=n-r$$

Theorem (1.2) Let M be an F-structure satisfying (1.1). Then

(1.4) (a) $$lF=Fl=F,\qquad mF=Fm=0$$

(b) $$F^{p_1p_2-1}\,l=-l,\qquad F^{p_1p_2-1}m=0$$

Proof: From (1.1), (1.2), (1.3)(a), (b), we get the results.

From (1.4) (b), it is clear that $F^{(p_1p_2-1)/2}$ acts on D_l as an almost complex structure and on D_m as a null operator.

Lakhan Singh, Department of Mathematics, D.J. College, Baraut, Baghpat (U.P.)
Shailendra Kumar Gautam, Eshan College of Engineering, Mathura(UP)

II. Nijenhuis Tensor:

Definifion (2.1) Let X and Y be any two vector fields on M, then their Lie bracket $[X, Y]$ is defined by

(2.1) $[X, Y] = XY- YX$,

and Nijenhuis tensor $N(X,Y)$ of F is defined as

(2.2)
$$N(X,Y)=[FX,FY]-F[FX,Y]-F[X,FY]+F^2[X,Y]$$

Theorem (2.1) *A* necessary and sufficient condition for the F-structure to be integrable is $N(X,Y)=0$, for any two vector fields X & Y on M.

Theorem (2.2) Let the F-structure satisfying (1.1) be integrable, then

(2.3)
$$\left(-F^{p_1p_2-2}\right)\left([FX,FY]+F^2[X,Y]\right)=l\left([FX,Y]+[X,FY]\right).$$

Proof: using theorem (2.1) in (2.2), we get

(2.4)
$$[FX,FY]+F^2[X,Y]=F\left([FX,Y]+[X,FY]\right)$$

Operating by $\left(-F^{p_1p_2-2}\right)$ on both the sides of (2.4) and using (1.2), we get the result.

Theorem (2.3) On the F-structure satisfying (1.1)

(2.5) (a) $$m\,N(X,Y)=m[FX,FY]$$

(b)
$$m\,N\left(F^{p_1p_2-2}X,Y\right)=m\left[F^{p_1p_2-1}X,FY\right]$$

Proof: Operating m on both the sides of (2.2) and using (1.4) (a) we get (2.5) (a). Replacing X by $F^{p_1p_2-2}X$ in (2.5) (a), we get (2.5) (b).

Theorem (2.4): On the F-structure satisfying (1.1), the following conditions are all equivalent

(2.6) (a) $$m\,N(X,Y)=0$$

(b) $$m[FX,FY]=0$$

(c) $$m\,N\left(F^{p_1p_2-2}X,Y\right)=0$$

(d) $$m\left[F^{p_1p_2-1}X,FY\right]=0$$

(e) $$m\left[F^{p_1p_2-1}lX,FY\right]=0$$

Proof: Using (1.4) (a), (b) in (2.5) (a), (b), we get the results.

III. CR-STRUCTURE:

Definiton (3.1) Let $T_c(M)$ denotes the complexified tangent bundle of the differentiable manifold *M*. A CR-structure on *M* is a complex sub-bundle *H* of $T_c(M)$ such that

(3.1) (a) $H_p \cap \tilde{H}_p = \{0\}$

(b) H is involutive that is $X, Y \in H \Rightarrow [X, Y] \in H$ for complex vector fields *X* and *Y*.

For the integrable F-structure satisfying (1.1) rank $((F)) = r = 2m$ on *M*.

we define

(3.2) $H_p = \{X - \sqrt{-1}FX : X \in X(D_l)\}$

where $X(D_l)$ is the $F(D_m)$ module of all differentiable sections of D_l.

Theorem (3.1) If *P* and *Q* are two elements of *H*, then

(3.3)

$$[P,Q] = [X,Y] - [FX,FY] - \sqrt{-1}(-1)([FX,Y] + [X,FY])$$

Proof: Defining $P = X - \sqrt{-1}(-1)FX, Q = Y - \sqrt{-1}(-1)FY$ and simplifying, we get (3.3)

Theorem (3.2) for $X, Y, \in X(D_l)$

(3.4)

$$l([FX,Y] + [X,FY]) = [FX,Y] + [X,FY]$$

Proof: Using (1.4) (a) and (2.1), we get the result as

(3.5)

$$\begin{aligned} l([FX,Y] + [X,FY]) &= l(FXY - YFX + XFY - FYX) \\ &= FXY - YFX + XFY - FYX \\ &= [FX,Y] + [X,FY] \end{aligned}$$

Theorem (3.3) The integrable F-structure satisfying (1.1) on *M* defines a CR-structure *H* on it such that

(3.6) $R_e(H) = D_l$

Proof: since $[X,FY], [FX,Y] \in X(D_l)$

then from (3.3), (3.4), we get

(3.7) $l[P,Q] = [P,Q]$

$\Rightarrow [P,Q] \in X(D_l)$

Thus *F* structure satisfying (1.1), defines a CR-structure on *M*.

Definition (3.2) Let $\tilde{K}$ be the complementary distribution of $R_e(H)$ to *TM*. We define a morphism $F : TM \longrightarrow TM$, given by

$F(X) = 0, \forall X \in X(\tilde{K})$ such that

(3.8) $F(X) = \frac{1}{2}\sqrt{-1}(-1)(P - \tilde{P})$

where $P = X + \sqrt{-1}(-1)Y \in X(H_p)$

and $\tilde{P}$ is complex conjugate of *P*.

Corollary (3.1): From (3.8) we get

(3.9) $F^2 X = -X$

Theorem (3.4): If *M* has CR-structure then $F^{p_1p_2} + F = 0$ and consequently F-structure satisfying (1.1) is defined on *M* s.t. D_l and D_m concide with $R_e(H)$ and $\tilde{K}$ respectively.

Proof: Since p_1 and p_2 are twin primes $\therefore$ $p_1 p_2$ when divided by 4 leaves 3 as a remainder $\therefore$ Repeated application of (3.9) gives,

$$\begin{aligned} F^{p_1p_2} &= F^3(X) \\ &= F(F^2X) \\ &= F(-X) \end{aligned}$$

Thus, $F^{p_1p_2} + F = 0$

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